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Abstract. The retarded Josephson equation, an integro-differential equation, will be solved for an exponential kernel including a trigonometric function (Dirichlet kernel). The characteristic and the limit cycles will be discussed. Especially, the critical limit cycle for a simple exponential kernel has been investigated.

1. Introduction

The full dynamics of a Josephson junction is given by the Werthamer equation (Werthamer 1966), which is an integro-differential equation. The retarded Josephson equation (Schlup 1975) can be considered as a mathematical simplification, which is at least as good as any differential approach to Werthamer's equation, i.e. its adiabatic approximation or linearised versions thereof.

For a point junction with current input, the retarded Josephson equation in dimensionless units is

$$\beta\ddot{\phi} + \dot{\phi} + \int_0^\infty \mathrm{d}t'\, F(t') \sin\phi(t-t') = \alpha, \tag{1.1}$$

where

$$\beta = \hbar C j_{\mathrm{m}}/2eG_{\mathrm{N}}^2, \qquad \alpha = I/j_{\mathrm{m}}, \qquad \int_0^\infty \mathrm{d}t\, F(t) = 1$$

and C = capacitance, G_{N} = normal conductance, j_{m} = maximal pair current and I the input current. The instantaneous voltage is $V = j_{\mathrm{m}}\hbar\dot{\phi}/2eG_{\mathrm{N}}$ and the unilateral Fourier transforms define the tunnel functions

$$j(\omega) + \mathrm{i}i(\omega) = \int_0^\infty \mathrm{d}t\, F(t)\, \mathrm{e}^{\mathrm{i}\omega t}. \tag{1.2}$$

Compared to Werthamer's equation, (1.1) assumes a linear quasi-particle term and contains the completely retarded pair term.

It is well known that the kernel $F(t)$ being an oscillatory (frequency ω_{g}) weakly ($\sim 1/t$) decreasing function is the reason that direct methods like Runge–Kutta combined with Gauss integration give only very crude results. On the other hand, the current–voltage characteristic for a rotational steady state can be evaluated by Fourier methods for not too small average voltages $\langle V\rangle$ with $\omega = \langle\dot{\phi}\rangle \gtrsim \omega_{\mathrm{g}}/5$ only. For smaller frequencies, too many higher harmonics have to be taken into account for a given precision of ω. Therefore, the critical current $\alpha_{\mathrm{c}}(\beta) = \alpha(\omega, \beta)|_{\omega=0}$ can never be determined. The same holds for the critical limit cycle, which is identical to the separatrix in a $\dot{\phi}$, ϕ phase plane for second-order differential equations. This separatrix gives insight into the variety of transient and steady-state solutions of the problem, since only one solution goes through a given point. This holds for higher-order differential equations (Hirsch and Smale 1974, Reissig *et al* 1969) if higher-dimensional phase spaces are used and in a certain limit sense also for integral equations, which can be considered as being differential of infinite order.

Bifurcation theory (Marsden and McCracken 1976) is the mathematical tool which handles problems concerned with the critical limit cycles. The fact that the solution bifurcates for certain values of the parameters means physically, for example, a unique steady-state solution becomes double or even multiple. Which of the solutions will be obtained depends on the switching process necessary to change the parameters. The point $\alpha_c(\beta)$ in the characteristic $\alpha(\omega, \beta)$ is such a bifurcation between the stationary solution and a rotational solution appearing for $\alpha > \alpha_c(\beta)$. As a result, the characteristic has a hysteresis which describes the selection of solutions chosen by the system, if parameters are changed quasistatically.

Questions of bifurcation are of great importance for the voltage resetting problem, i.e. the sudden fall-back of the rotational to the stationary solution if the current is decreased slowly below some value $\alpha_R(\beta)$. Qualitatively, it can be understood as a discontinuous voltage jump from the absolute current minimum, if it occurs for a finite voltage $\omega_R(\beta)$. If the absolute α minimum is in $\alpha_c(\beta)$ no deterministic resetting occurs (see noise resetting in Falco 1974).

For a numerical evaluation of the critical limit cycles, the integral has to decay rather fast. If the kernel $F(t)$ is a finite Dirichlet series, the integral equation (1.1) can be transferred into a higher-order differential equation (Schlup 1975), for which direct time integration by Runge–Kutta or similar methods is possible.

Here we investigate some properties of the critical limit cycles for third- and fourth-order differential-type retarded Josephson equations.

2. The exponential kernel

The Dirichlet kernel considered is

$$F(t) = N\,\mathrm{e}^{-\lambda t}\cos(\Omega t + \delta), \tag{2.1}$$

where normalisation gives $N = (\lambda^2 + \Omega^2)/(\lambda\cos\delta - \Omega\sin\delta)$. The corresponding tunnel functions are

$$\begin{aligned} j(\omega) &= \frac{N}{2}\left(\frac{\lambda\cos\delta - (\omega+\Omega)\sin\delta}{\lambda^2 + (\omega+\Omega)^2} + \frac{\lambda\cos\delta + (\omega-\Omega)\sin\delta}{\lambda^2 + (\omega-\Omega)^2}\right) \\ i(\omega) &= \frac{N}{2}\left(\frac{\lambda\sin\delta + (\omega+\Omega)\cos\delta}{\lambda^2 + (\omega+\Omega)^2} + \frac{-\lambda\sin\delta + (\omega-\Omega)\cos\delta}{\lambda^2 + (\omega-\Omega)^2}\right). \end{aligned} \tag{2.2}$$

Elimination of the integral in (1.1) gives the differential equation of fourth order

$$\frac{\beta}{\lambda^2+\Omega^2}\phi^{(4)} + \frac{2\lambda\beta+1}{\lambda^2+\Omega^2}\phi^{(3)} + \left(\beta + \frac{2\lambda}{\lambda^2+\Omega^2}\right)\ddot{\phi} + \dot{\phi}\left(1 + \frac{\cos\delta}{\lambda\cos\delta - \Omega\sin\delta}\cos\phi\right) + \sin\phi = \alpha \tag{2.3}$$

or for $F = \lambda\, e^{-\lambda t}$, i.e. $\Omega = \delta = 0$,

$$\frac{\beta}{\lambda^2}\phi^{(4)} + \left(\frac{2\beta}{\lambda} + \frac{1}{\lambda^2}\right)\phi^{(3)} + \left(\beta + \frac{2}{\lambda}\right)\ddot{\phi} + \dot{\phi}\left(1 + \frac{1}{\lambda}\cos\phi\right) + \sin\phi = \alpha \tag{2.4}$$

which is equivalent to the third-order equation

$$\frac{\beta}{\lambda}\phi^{(3)} + \left(\beta + \frac{1}{\lambda}\right)\ddot{\phi} + \dot{\phi} + \sin\phi = \alpha, \tag{2.5}$$

since

$$\text{equation (2.4)} \equiv \text{equation (2.5)} + \frac{1}{\lambda}\,\text{equation (2.5)} \tag{2.6}$$

and the homogeneous solution of (2.6), which may be added to (2.5) vanishes for $t \to +\infty$; i.e. the transient solutions of (2.4) and (2.5) may be different but the steady-state solutions are the same. The form (2.5) also follows directly from (1.1) by differentiation. The tunnel functions for $\Omega = \delta = 0$ are

$$j(\omega) = \frac{\lambda^2}{\lambda^2 + \omega^2}, \qquad i(\omega) = \frac{\lambda\omega}{\lambda^2 + \omega^2}. \tag{2.7}$$

The adiabatic approximation to the retarded Josephson equation would be

$$\beta\ddot{\phi} + \dot{\phi} - i(\dot{\phi})\cos\phi + j(\dot{\phi})\sin\phi = \alpha, \tag{2.8}$$

which for small $\dot{\phi}$ (linearised adiabatic approximation) becomes the classical Josephson equation

$$\beta\ddot{\phi} + \dot{\phi}(1 + \gamma\cos\phi) + \sin\phi = \alpha \tag{2.9}$$

with

$$-\gamma = \bar{t}^F = \int_0^\infty dt\, t\, F(t) = [(\lambda^2 - \Omega^2)\cos\delta - 2\lambda\Omega\sin\delta]/[(\lambda^2 + \Omega^2)(\lambda\cos\delta - \Omega\sin\delta)].$$

For $\lambda \to \infty$ the kernel is a $\delta(t-0)$ function giving the circuit model Josephson equation

$$\beta\ddot{\phi} + \dot{\phi} + \sin\phi = \alpha \tag{2.10}$$

exhibiting no ϕ-dependent dissipative term. For $\lambda \to 0$, the dominant terms in (2.3) ((2.5)) give

$$\beta\phi^{(4)} + \phi^{(3)} = 0 \qquad \beta\phi^{(3)} + \ddot{\phi} = 0,$$

which, for a general rotational solution ($\dot{\phi}$ = periodic function of ωt with period 2π) means $\ddot{\phi}, \phi^{(3)}, \ldots = 0$ in a steady state. Therefore, $\dot{\phi} = \text{const} = \omega$, and $\alpha = \omega$ is the characteristic for $\lambda = 0$.

3. The characteristic

The tunnel functions with $\Omega = 1$ have been chosen to be very similar to the functions resulting from BCS theory. For $\lambda = 1$, $\delta = 0$ the $j(\omega)$ is positive everywhere with a maximum in $\omega = 1 (= \Omega)$ and decreasing for $\omega \to \infty$. For small ω, $i(\omega)$ behaves like ω^3, and goes through a maximum to zero for $\omega \to \infty$ in rough agreement with a zero-temperature case. The $j(\omega)$ maximum becomes more pronounced and $i(\omega) \approx -\text{const}\,\omega$, if either δ is made positive $<\pi/4$, or if λ is decreased but still $>1/2$ in qualitative agreement with the low-temperature case.

The average voltage–current characteristic $\omega(\alpha, \beta, \lambda, \Omega, \delta)$ is determined for $\beta = 2$ in figure 1 for both kernels (equations (2.3) and (2.5)) together with the linearised adiabatic approximation (LAA). All the curves have hyperbolic-like shapes and deviate appreciably from the LAA especially for large ω. For the oscillating kernel, there is no special structure at $\omega = \Omega/n$ like the subharmonic gaps for the Werthamer equation, at least not for the parameters chosen. This may result from the singularity-free tunnel functions, as a consequence of an exponential kernel. The characteristic for $\lambda = 0$ and $\lambda = \infty$ is also plotted. The adiabatic approximation resulting from a solution of equation (2.8) is the chain curve, which lies in between the LAA for $\gamma = 0$ and -1, since $i(\omega)/\omega$ is, on the average, smaller than 1 for $\lambda = 1$, and ω being of order 1. It agrees better than the LAA for large ω if compared with the characteristic of the retarded Josephson equation, since the LAA is a further simplification of (2.8), and since the instantaneous voltage changes only a little for large ω.

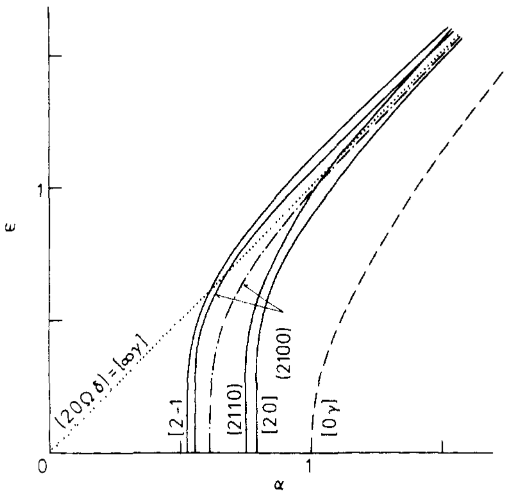

Figure 1. The characteristics $\omega(\alpha)$ for the kernel $F(t) = N(\exp - \lambda t)\cos(\Omega t + \delta)$ marked with $(\beta, \lambda, \Omega, \delta)$ or with $[\beta, \gamma]$ for the classical Josephson equation. The chain curve is the adiabatic approximation (equation (2.8)).

4. The critical limit cycles

The differential equation (2.3) is most effectively solved by integration in the phase plane variables $z = \dot{\phi}$ and $x = \phi - \pi + \sin^{-1}\alpha$ for $|\alpha| \leqslant 1$ or ϕ otherwise. Using $\dot{\phi} = z(\phi)$, $\ddot{\phi} = z'\dot{\phi} = zz'$, $\phi^{(3)} = (zz')'\dot{\phi} = z(zz')'$, etc, (2.3) transforms into

$$\frac{\beta}{\lambda^2+\Omega^2} z(z(zz')')' + \frac{2\lambda\beta+1}{\lambda^2+\Omega^2} z(zz')' + \left(\beta + \frac{2\lambda}{\lambda^2+\Omega^2}\right) zz'$$

$$+ z\left(1 + \frac{\cos\delta}{\lambda\cos\delta - \Omega\sin\delta}\cos\phi\right) + \sin\phi = \alpha, \tag{4.1}$$

and (2.5) into

$$\frac{\beta}{\lambda} z(zz')' + \left(\beta + \frac{1}{\lambda}\right) zz' + z + \sin\phi = \alpha. \tag{4.2}$$

The following boundary-value problem has to be solved:

$$z(2\pi) = z(0), \tag{4.3}$$

and subsequently,

$$\omega = 2\pi \left(\int_0^{2\pi} \frac{d\phi}{z(\phi)} \right)^{-1} \tag{4.4}$$

gives the characteristic. In order to find $\phi(t)$, a further integration is required:

$$t = \int_0^{\phi} \frac{d\bar{\phi}}{z(\bar{\phi})} \tag{4.5}$$

if $\phi(0) = 0$ is assumed.

It can be shown that the period $2\pi/\omega$ increases if $\alpha \gtrsim \alpha_c$ because of a slow motion near to the unstable equilibrium point $\phi = \bar{\phi}_\alpha = \pi - \sin^{-1}\alpha$ being infinite for $\alpha = \alpha_c$ (see appendix). For $\alpha = \alpha_c$, equations (4.1) and (4.2) are singular since $\dot{\phi}(\phi = \bar{\phi}_\alpha) = z(0) = 0$. To start numerical integration, the derivatives $z'(0)$, $z''(0)$ and for (4.1) $z'''(0)$ are required. They can be determined from (4.1) and (4.2), if a power expansion for $z(x)$ is used and if α_c is known.

Figure 2 shows the limit cycles in the $\dot{\phi}$, ϕ plane for $\alpha \gtrsim \alpha_c$ and equation (2.3) with $\beta = 2$, $\lambda = \Omega = 1$, $\delta = 0$. There exist two limiting tangents and the separatrix looks similar to that of the classical Josephson equation (2.9).

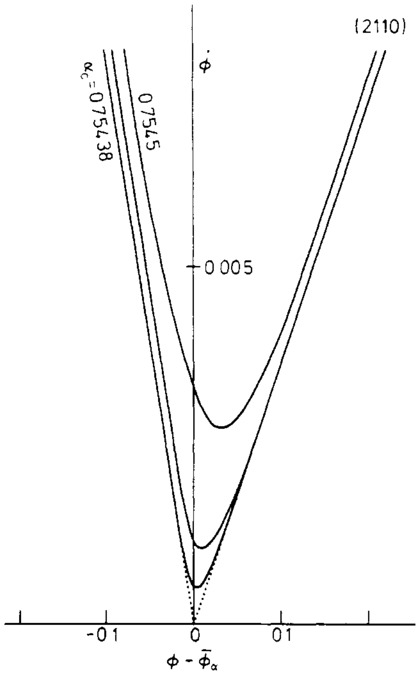

Figure 2. The limit cycles near the unstable equilibrium $\bar{\phi}_\alpha$ for (2 1 1 0) and $\alpha \gtrsim \alpha_c = 0{\cdot}754\ 38$ in a reduced phase space.

In contrast, figure 3 shows the same for equation (2.5) with $\beta = 2$, $\lambda = 1$ for $\alpha \gtrsim \alpha_c$. It can be seen how a loop in the phase projection develops; it indicates that the motion inverts its direction just behind the unstable equilibrium. The outgoing branch follows $\dot\phi \approx m(\phi - \bar\phi_\alpha)$ for a large angular range, whereas the incoming branch behaves spiral-like giving rise to a focal point. This is shown in figure 4 which gives the phase space for equation (2.5). The curve drawn is an exponential spiral near to the origin, but since it deçays very fast only half of a winding can be seen in this plot.

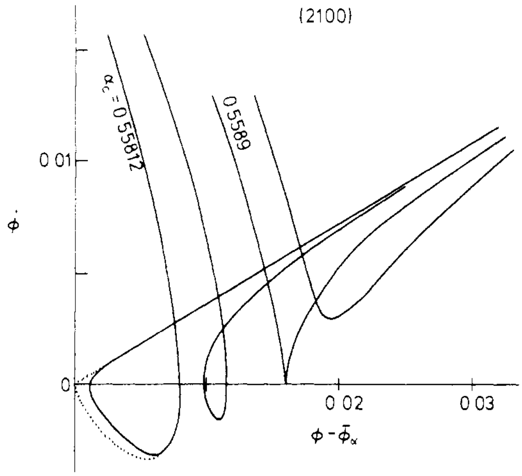

Figure 3. The limit cycles near the unstable equilibrium $\bar\phi_\alpha$ for (2 1 0 0) and $\alpha \gtrsim \alpha_c = 0{\cdot}558\ 12$ in a reduced phase space.

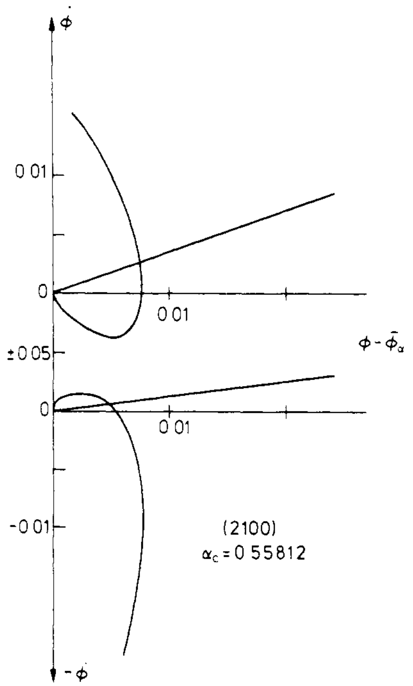

Figure 4. The critical limit cycle for (2 1 0 0) near to $\phi = \bar\phi_\alpha$. The phase space ϕ, $\dot\phi$, $\ddot\phi$ is given by two projections. The incoming branch is spiral-like.

5. Final remarks

As a result of numerical problems, the critical limit cycle of the retarded Josephson equation can only be found by fast-decaying kernels or by kernels consisting of a finite Dirichlet sum. As the simplest Dirichlet kernel, the exponential function with or without a trigonometric one is investigated. Contrary to a weakly-decaying oscillating kernel, the characteristic shows no drastic structure if the solution is in resonance with the trigonometric function, because of its smooth Fourier transforms. The critical limit cycle (separatrix) then behaves as in the classical Josephson equation, generating a saddle point or a focus.

If the kernel is a simple exponential, the critical limit cycle behaves linearly and with finite tangent only in its outgoing branch. The incoming branch is spiral-like close to the unstable equilibrium point giving rise to a focus.

The characteristics are different for the retarded equation, the adiabatic approximation and their linear version, since for $\beta = 2$ the voltage changes rather fast. Even in the limit $\omega \to 0$, the critical currents are different, since the voltage still changes in a range of order 1. A coincidence of the three results can only be expected for $\beta \gg 1$, if the tunnel functions have a Taylor series for small ω. This can be proved by asymptotic expansions with respect to $1/\beta$.

Appendix. Motion near to the unstable equilibrium $\bar{\phi}_\alpha$

Assuming $x = \phi - \bar{\phi}_\alpha$ and $z = mx + \mathrm{O}(x^2)$ for the separatrix near to the unstable equilibrium in the phase plane z, x (reduced phase space)

$(1/x)$ equation (4.1) for $\alpha = \alpha_c (<1) \Rightarrow$

$$P_4(m) = \frac{\beta}{\lambda^2+\Omega^2} m^4 + \frac{2\beta\lambda+1}{\lambda^2+\Omega^2} m^3 + \left(\beta + \frac{2\lambda}{\lambda^2+\Omega^2}\right) m^2$$
$$+ \left(1 - \frac{\cos\delta}{\lambda\cos\delta - \Omega\sin\delta}(1-\alpha_c^2)^{1/2}\right) m - (1-\alpha_c^2)^{1/2} = 0. \tag{A.11}$$

Analogously

$$(4.2) \Rightarrow P_3(m) = \frac{\beta}{\lambda} m^3 + \left(\beta + \frac{1}{\lambda}\right) m^2 + m - (1-\alpha_c^2)^{1/2} = 0, \tag{A.2}$$

which is a special case of (A.1) for $\Omega = \delta = 0$:

$$P_4(m)|_{\Omega=\delta=0} = P_4^*(m) = \left(\frac{m}{\lambda} + 1\right) P_3(m) = 0. \tag{A.3}$$

For $\beta = 2$, $\lambda = 1$, $\Omega = 1$, $\delta = 0$ with $\alpha_c = 0{\cdot}754\,38$ (A.1) $\Rightarrow m_1 = 0{\cdot}362\,86$, $m_2 = -0{\cdot}728\,18$ and conjugate complex $m_{3,4} = -1{\cdot}067\,34 \pm \mathrm{i}\,1{\cdot}159\,81$.

For $\beta = 2$, $\lambda = 1$, $\Omega = 0$, $\delta = 0$ with $\alpha_c = 0{\cdot}558\,12$ (A.2) $\Rightarrow m_1^0 = 0{\cdot}356\,85$ and conjugate complex $m_{2,3}^0 = -0{\cdot}928\,43 \pm \mathrm{i}\,0{\cdot}548\,31$, whereas $P_4^*(m)$ has the additional root $m = -1$, which is a consequence of (2.6) and has to be discarded in a steady state.

The time-dependent behaviour near to the unstable equilibrium can be discussed by transforming one higher-order equation (2.3) or (2.5) into a system of first-order equations, which can be linearised for small deviations $\phi_1 = \phi - \bar{\phi}_\alpha$. Introducing

$\phi_2 = \dot{\phi}$, $\phi_3 = \ddot{\phi}$ and eventually $\phi_4 = \phi^{(3)}$ and using a column vector $\boldsymbol{\phi}$ with components ϕ_i the differential equation becomes

$$\dot{\boldsymbol{\phi}} = \mathbf{A}\boldsymbol{\phi}, \tag{A.4}$$

where $\mathbf{A}$ is a matrix with eigenvalues being the roots m_i of (A.1) for (2.3) or being the roots m_i^0 of (A.2) for (2.5). Since the roots are all different from one another, there exists a regular matrix $\mathbf{S}$ diagonalising $\mathbf{A}$

$$\mathbf{M} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & \ddots \end{pmatrix}. \tag{A.5}$$

Therefore the solution near to $\phi_1 \simeq 0$ is

$$\boldsymbol{\phi} = \mathbf{S}\boldsymbol{C}\, \mathrm{e}^{\mathbf{M}t}, \tag{A.6}$$

where $\boldsymbol{C}$ is a vector of integration contants.

For (2.3) the outgoing branch starting with $t = -\infty$ in the unstable equilibrium is given by $\phi_1 = C_1 \exp(m_1 t)$, $c_2 = c_3 = c_4 = 0$, whereas the incoming branch ending with $t = \infty$ in $\bar{\phi}_\alpha$ is given by $c_1 = 0$ and $\phi_1 \approx c_2 \exp(m_2 t)$, since m_2 with $|m_2| < |\mathrm{Re}\, m_{3,4}|$ dominates for large times in agreement with the numerical results.

For (2.5) the outgoing branch is analogously given by $\phi_1 = c_1\, (\exp m_1^0 t)\; c_2 = c_3 = 0$ starting in $\bar{\phi}_\alpha$ with $t = -\infty$. The incoming branch ending in $t = \infty$ however is determined by $c_1 = 0$, $\phi_1 = c_2 \exp[(\mathrm{Re}\, m_2^0)t] \cos[(\mathrm{Im}\, m_2^0)t + c_3]$, $\phi_2 = \dot{\phi}_1$ and $\phi_3 = \ddot{\phi}_1$, which is a spiral with half period $T_{1/2} = \pi/\mathrm{Im}\, m_2^0$ during which ϕ_1, ϕ_2, ϕ_3 change sign and decrease by $-\phi_i(t + T_{1/2})/\phi_i(t) = \exp[(\mathrm{Re}\, m_2^0)T_{1/2}]$. For the example of (A.2) considered, this factor is 1:200 for a half period or 1:40000 for a full spiral period. It is easy to verify that the spiral lies in the plane

$$|m_2^0|^2\phi_1 - 2(\mathrm{Re}\, m_2^0)\phi_2 + \phi_3 = 0.$$

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